Appendix: Proofs

In what follows, there are several useful, alternative ways of expressing the worker's utility when employed, given here for reference (for the model with no loss aversion and no money illusion):

$$\begin{aligned} U^E(w,e) &= \sigma u(w,e) + (1-\sigma) f(\tilde{u}(w,e), \tilde{\pi}(w,e;p)) \\ &= \mu\tilde{u} + (1-\mu)\tilde{\pi} + \sigma u_0 \\ &= (1-\mu)pe - \mu c(e) + (2\mu - 1)w - (\mu-\sigma)u_0 - (1-\mu)\pi_0, \end{aligned}$$

where $\mu \in \{\mu_D, \mu_A\}$.

Lemma 1. *Under Assumption A, for any $p > 1$, there exists $\overline{w}(p)$ such that:*

1. *If $w \leq \overline{w}(p)$, then $e(w,p) \equiv \tilde{e}(p,w)$. Moreover, $\tilde{e}(p,w)$ is increasing in w and decreasing in p.*
2. *If $w > \overline{w}(p)$, then $e(w,p) \in [e_{\text{eff}}(p), \tilde{e}(w,p))$, and $e(w,p)$ is constant in w and increasing in p.*

Proof: Note that $\tilde{\pi}(w,e;p) \leq \tilde{u}(w,e)$ is equivalent to

$$c(e) + pe \leq 2w + \pi_0 - u_0. \tag{1}$$

Furthermore, the left-hand side is strictly increasing in e since $c' > 0$ and $p > 1$. Therefore, $\tilde{\pi}(w,e;p) \leq \tilde{u}(w,e)$ is equivalent to $e \leq \tilde{e}$.

The worker's effort level solves $e(w,p) = \arg\max_e U^E(w,e;p)$. Since U^E is kinked at $\tilde{u} = \tilde{\pi}$, which corresponds to $e = \tilde{e}$, we have

$$\max_e U^E(w,e;p) \equiv \max \Big\{ \underbrace{\max_{e<\tilde{e}} (1-\mu_A)pe - \mu_A c(e)}_{\text{(I)}}, \underbrace{\max_{e \geq \tilde{e}} (1-\mu_D)pe - \mu_D c(e)}_{\text{(II)}} \Big\}.$$

First consider sub-problem (II), and define e_{II} to be the solution to the sub-problem. Assumption A(i) states $\mu_D \geq 1$. Then, on $e \geq \tilde{e}$,

$$\frac{\partial U^E}{\partial e} = (1-\mu_D)p - \mu_D c'(e) < 0.$$

That is, since $\mu_D \geq 1$, U^E is strictly decreasing in e for all $e \geq \tilde{e}$. Hence the maximum is achieved at $e_{\mathrm{II}} = \tilde{e}$.

Consider maximization problem (I) and denote by e_{I} the solution to this sub-problem. In particular, $e_{\mathrm{I}} > 0$ is given by the first-order condition

$$c'(e_{\mathrm{I}}) = p\frac{1-\mu_A}{\mu_A} \geq p \tag{2}$$

since A(ii) implies $\frac{1-\mu_A}{\mu_A} \geq 1$. (Since $\frac{\partial^2 U^E}{\partial e^2} = -\mu_A c''(e)$, $c'' > 0$, and $\mu_A > 0$, the second-order condition for maximization is satisfied on $e > 0$.)

Define $\overline{w}(p)$ by

$$\tilde{e}(p, \overline{w}(p)) = e_{\mathrm{I}}. \tag{3}$$

Since $\tilde{e}$ is increasing in w (see below), for all $w \leq \overline{w}(p)$ we have $e_{\mathrm{I}} \geq \tilde{e}$, and so (I) does not have a solution (because the objective function is strictly increasing in e for all $e < \tilde{e}$). Therefore, if $w \leq \overline{w}(p)$, then the worker's effort level is the solution from (II): $e(w,p) = \tilde{e}$. If $w > \overline{w}(p)$, then (I) has solution e_{I}, which gives higher utility than the solution to (II), $\tilde{e}$, since

$$(1-\mu_A)pe_{\mathrm{I}} - \mu_A c(e_{\mathrm{I}}) > (1-\mu_A)p\tilde{e} - \mu_A c(\tilde{e}) > (1-\mu_D)p\tilde{e} - \mu_D c(\tilde{e}),$$

where the first inequality follows from e_{I} being the solution to (I), and the second inequality follows from $\mu_D > \mu_A$. Therefore, $e(p,w) = e_{\mathrm{I}}$.

Summarizing,

$$e(w,p) = \begin{cases} \tilde{e}(w,p) & \text{if } w \leq \overline{w}(p) \\ e_{\mathrm{I}}(p) & \text{if } w > \overline{w}(p). \end{cases}$$

Equation (2) shows that e_{I} is constant in w and increasing in p. By the definition of $\overline{w}(p)$, $e_{\mathrm{I}}(p) = \tilde{e}(w,p)$ when $w = \overline{w}(p)$, and since $\tilde{e}$ is increasing in w, $e_{\mathrm{I}} < \tilde{e}$ when $w > \overline{w}(p)$. Note that equation (2), A(ii), and $w > \overline{w}(p)$ imply that $e_{\mathrm{I}} \geq e_{\mathrm{eff}}(p)$. Putting these inequalities together: when $w > \overline{w}(p)$, $e_{\mathrm{I}}(p) \in [e_{\mathrm{eff}}(p), \tilde{e}(w,p))$.

Finally, analysis of equation (2) when it holds with equality shows that if we increase w, we must also increase e to achieve equality; i.e., $\frac{\partial \tilde{e}(w,p)}{\partial w} > 0$. Similarly, increasing p increases the left-hand side, so e must be decreased to keep equality; i.e., $\frac{\partial \tilde{e}(w,p)}{\partial p} < 0$.

□

Proposition 1. *Under Assumptions A and B, for any $p > 1$, there is a unique equilibrium in which the firm hires the worker, and the equilibrium transaction (w^*, e^*) satisfies $\pi(w^*, e^*; p) - \pi_0 = u(w^*, e^*) - u_0$ and $e^* = e_{\mathrm{eff}}(p)$.*

Proof: Recall from Lemma 1 that

$$e(w,p) = \begin{cases} \tilde{e}(w,p) & w \leq \overline{w}(p) \\ e_{\mathrm{I}}(p) & w > \overline{w}(p). \end{cases}$$

Therefore, the firm's maximization problem is

$$\max_w \pi(w, e(w,p); p) = \max\Big\{ \underbrace{\max_{w \leq \overline{w}(p)} p\tilde{e}(w,p) - w}_{\text{(A)}}, \underbrace{\max_{w > \overline{w}(p)} pe_{\mathrm{I}}(p) - w}_{\text{(B)}}, 0 \Big\}$$

where (A) and (B) are subject to the employment constraint, $U^E(w,e) \geq 0$. The 0 corresponds to the firm's outside option. Let w_{A} and w_{B} denote the solutions to (A) and (B), respectively.

For sub-problem (A), the first-order condition implies

$$\frac{\partial \tilde{e}(w_{\mathrm{A}},p)}{\partial w} = \frac{1}{p}.$$

Moreover, implicitly differentiating equation (2) with equality with respect to w, we have

$$c'(\tilde{e}(w,p))\frac{\partial \tilde{e}(w,p)}{\partial w} + p\frac{\partial \tilde{e}(w,p)}{\partial w} = 2.$$

Combining, we have $c'(\tilde{e}(w_{\mathrm{A}},p)) = p$. That is, $\tilde{e}(w_{\mathrm{A}},p) = e_{\mathrm{eff}}(p)$. To see that the second-order condition for a global maximum is satisfied, note that an increase in w increases $\tilde{e}$, so $c'(\tilde{e})$ also increases. To maintain the equality, it must be that $\frac{\partial \tilde{e}}{\partial w}$ decreases, hence $\frac{\partial^2 \tilde{e}}{\partial w^2} < 0$. It follows that $\frac{\partial^2 \pi}{\partial w^2} = p\frac{\partial^2 \tilde{e}}{\partial w^2} < 0$. Therefore, the unique solution to (A) is given by $e_{\mathrm{A}} = \tilde{e}(w_{\mathrm{A}},p) = e_{\mathrm{eff}}(p)$ and w_{A} such that $\tilde{u}(w_{\mathrm{A}}, e_{\mathrm{A}}) = \tilde{\pi}(w_{\mathrm{A}}, e_{\mathrm{A}}; p)$, which implies

$$w_{\mathrm{A}} = \frac{c(e_{\mathrm{eff}}(p)) + pe_{\mathrm{eff}}(p) - \pi_0 + u_0}{2}. \tag{4}$$

We now check that both players choose to interact rather than taking their outside options. The worker chooses employment whenever $U^E(w_{\mathrm{A}}, e_{\mathrm{eff}}(p)) \geq 0$. Since

$$\begin{aligned} U^E(w_{\mathrm{A}}, e_{\mathrm{eff}}(p)) &= \sigma u(w_{\mathrm{A}}, e_{\mathrm{eff}}) + (1-\sigma)\left(u(w_{\mathrm{A}}, e_{\mathrm{eff}}) - u_0\right) \\ &= w_{\mathrm{A}} - c(e_{\mathrm{eff}}) - (1-\sigma)u_0 \\ &= \frac{c(e_{\mathrm{eff}}(p)) + pe_{\mathrm{eff}}(p) - \pi_0 + u_0}{2} - c(e_{\mathrm{eff}}) - (1-\sigma)u_0 \\ &= \frac{M(p)}{2} - \frac{\pi_0}{2} + \frac{2\sigma - 1}{2}u_0, \end{aligned}$$

accepting the offer is equivalent to $(2\sigma - 1)u_0 - \pi_0 \geq -M(p)$. On the other hand, the firm offers the wage only if $\pi(w_{\mathrm{A}}, e_{\mathrm{eff}}(p); p) \geq 0$. Since

$$\pi(w_{\mathrm{A}}, e_{\mathrm{eff}}(p)) = pe_{\mathrm{eff}}(p) - w_{\mathrm{A}} = \frac{pe_{\mathrm{eff}}(p) - c(e_{\mathrm{eff}}(p)) + \pi_0 - u_0}{2},$$

making the offer is equivalent to $u_0 - \pi_0 \leq M(p)$. Both conditions are implied by Assumption B.

We now check that the candidate solution is in fact an interior solution. Note that $w_{\mathrm{A}} \leq \overline{w}(p)$ is equivalent to $e_{\mathrm{I}} \geq \tilde{e}(w_{\mathrm{A}}, p) = e_{\mathrm{eff}}(p)$, which is guaranteed by A(ii) and $c'' > 0$ since e_{I} is defined by equation (2). Therefore (A) has an interior solution.

Clearly if (B) has no solution, then the solution to (A) is the solution to the maximization problem. So suppose (B) has a solution $w_B > \overline{w}(p)$. (Note that even though the maximand in (B) is strictly decreasing, a solution defined by the employment constraint $U^E = 0$ may exist.) We show that in this case, the solution to (A) dominates the solution to (B). Since w_A is the solution to (A), it must necessarily be the case that

$$p\tilde{e}(w_{\mathrm{A}}, p) - w_{\mathrm{A}} \geq p\tilde{e}(\overline{w}(p), p) - \overline{w}(p).$$

Since $\overline{w}(p)$ is defined by $\tilde{e}(\overline{w}(p), p) = e_{\mathrm{I}}$ in equation (3) and $w_B > \overline{w}(p)$, we have

$$p\tilde{e}(\overline{w}(p), p) - \overline{w}(p) = pe_{\mathrm{I}} - \overline{w}(p) > pe_{\mathrm{I}} - w_{\mathrm{B}},$$

Combining, we see that the solution to (A) dominates the solution to (B) whenever the latter exists.

□

Proposition 2. *At the equilibrium described in Proposition 1:*

1. *w^* and e^* are both increasing in p.*
2. *e^* does not depend on u_0 nor π_0.*
3. *w^* is increasing in u_0 and decreasing in π_0.*

Proof: From the proof of Proposition 1, we know that w^* is given by equation (4) and $e^* = e_{\mathrm{eff}}(p)$. Since $e_{\mathrm{eff}}(p)$ is defined by $c'(e_{\mathrm{eff}}(p)) = p$ and since $c'' > 0$, $e_{\mathrm{eff}}(p)$ is increasing in p and is constant in both u_0 and π_0. Given equation (4), w^* is clearly decreasing in π_0 and increasing in u_0. Furthermore, since $e_{\mathrm{eff}}(p)$ is increasing in p and since $c' > 0$, w^* is also increasing in p.

□

Proposition 3. *Under Assumptions A' and B', with positive probability there is a unique equilibrium in which the firm hires the worker in both periods. The equilibrium transactions, (w_t^*, e_t^*) for $t = 1, 2$, satisfy $\pi(w_t^*, e_t^*; p_t) - \pi_{t-1} = u(w_t^*, e_t^*) - u_{t-1}$ and $e_t = e_{\mathrm{eff}}(p_t)$.*

Proof: We proceed by backwards induction.

First, note that Assumption A is implied by Assumption A$'$, and there is positive probability that the realized value of p_2 satisfies Assumption B. In that case, the equilibrium in period 2 is given by Proposition 1, where the reference transactions, u_1 and π_1, are the period-1 outcomes. Since $M(p)$ is strictly increasing in p, we can define p^* as the smallest value of p that satisfies Assumption B for u_1 and π_1. Assuming from now on that $p_2 \geq p^*$, $e_2^* = e_{\text{eff}}(p_2)$ and $\pi(w_2^*, e_2^*; p_2) - \pi_1 = u(w_2^*, e_2^*) - u_1$, which imply that

$$w_2^* = \frac{c(e_{\text{eff}}(p_2)) + p_2 e_{\text{eff}}(p_2) - \pi_1 + u_1}{2}.$$

The resulting profit and utility are

$$\begin{aligned} \pi_2 &= p_2 e_2^* - w_2^* \\ &= p_2 e_2^* - \frac{c(e_{\text{eff}}(p_2)) + p_2 e_{\text{eff}}(p_2) - \pi_1 + u_1}{2} \\ &= \frac{M(p_2) + \pi_1 - u_1}{2} \end{aligned}$$

and

$$\begin{aligned} U_2^E &= \sigma u(w_2^*, e_2^*) + (1-\sigma) f(\tilde{u}(w_2^*, e_2^*), \tilde{\pi}(w_2^*, e_2^*)) \\ &= \sigma u(w_2^*, e_2^*) + (1-\sigma)\tilde{u}(w_2^*, e_2^*) \\ &= u(w_2^*, e_2^*) - (1-\sigma)u_1 \\ &= w_2^* - c(e_2^*) - (1-\sigma)u_1 \\ &= \frac{c(e_2^*) + pe_2^* - \pi_1 + u_1}{2} - c(e_2^*) - (1-\sigma)u_1 \\ &= \frac{M(p_2) - \pi_1 + (2\sigma - 1)u_1}{2}. \end{aligned}$$

Turning our attention to period 1, we first consider the worker's optimization problem. For $\mu \in \{\mu_A, \mu_D\}$, the worker's objective function is given by

$$\begin{aligned} U_1^E + EU_2^E &= \mu\tilde{u}_1 + (1-\mu)\tilde{\pi}_1 + \sigma u_0 + \frac{1}{2}EM(p_2) - \frac{1}{2}\pi_1 + \frac{2\sigma - 1}{2}u_1 \\ &= \frac{2\mu + 2\sigma - 1}{2}u_1 + \frac{1-2\mu}{2}\pi_1 - (\mu - \sigma)u_0 - (1-\mu)\pi_0 + \frac{1}{2}EM(p_2). \end{aligned}$$

Rescaling, and noting that w_1, u_0, π_0, and $EM(p_2)$ are constants (with respect to the maximization), the worker's maximization problem takes the form

$$\max_e U_1^E(w,e) + EU_2^E \equiv \max \Bigg\{ \underbrace{\max_{e<\tilde{e}} (1-2\mu_A)p_1 e - (2\mu_A + 2\sigma - 1)c(e)}_{\text{(I)}}, \underbrace{\max_{e\geq\tilde{e}} (1-2\mu_D)p_1 e - (2\mu_D + 2\sigma - 1)c(e)}_{\text{(II)}} \Bigg\}.$$

Consider sub-problem (II) first. Note that A′(i) implies $1 - 2\mu_D < 0$ and $2\mu_D + 2\sigma - 1 > 0$. Therefore, the worker's objective function is strictly decreasing on $e \geq \tilde{e}$. Thus, the solution to (II) is $e_{\text{II}} = \tilde{e}$.

For sub-problem (I), A′(ii) implies $1 - 2\mu_A \geq 0$. If $2\mu_A + 2\sigma - 1 \leq 0$ (i.e., $\mu_A \leq \frac{1-2\sigma}{2} < \frac{1}{2}$ since $\sigma > 0$), then the maximand of (I) is strictly increasing in e. In that case, (I) has no solution. On the other hand, if $2\mu_A + 2\sigma - 1 > 0$ (i.e., $\mu_A > \frac{1-2\sigma}{2}$), then the worker's first-order condition implies

$$c'(e_{\text{I}}) = \frac{1-2\mu_A}{2\mu_A + 2\sigma - 1} p_1. \tag{5}$$

(The second-order condition is satisfied in this case since $2\mu_A + 2\sigma - 1 > 0$ and $c'' > 0$.) We conclude that the solution to the worker's problem in period 1 is

$$e_1(w,p_1) = \begin{cases} e_{\text{I}} & \text{if } \mu_A > \frac{1-2\sigma}{2} \text{ and } w > \overline{w}(p_1) \\ \tilde{e} & \text{otherwise,} \end{cases}$$

where (as in the proof of Lemma 1) $\overline{w}(p)$ is defined by $\tilde{e}(p, \overline{w}(p)) = e_{\text{I}}$.

We now turn to the firm's period-1 maximization problem. The firm's profit is given by

$$\begin{aligned} \pi_1 + E\pi_2 &= \pi_1 + \frac{1}{2}EM(p_2) + \frac{1}{2}\pi_1 - \frac{1}{2}u_1 \\ &= \frac{3}{2}p_1 e_1 - 2w_1 + \frac{1}{2}c(e_1) + \frac{1}{2}EM(p_2). \end{aligned}$$

Again, since $EM(p_2)$ is a constant, we will drop this term from the maximization problems in the following.

Case 1. $\mu_A \leq \frac{1-2\sigma}{2}$. In this case, the worker's effort level conditional on accepting employment is $\tilde{e}$ (regardless of whether $w > \overline{w}(p_1)$ or $w \leq \overline{w}(p_1)$). Therefore, the firm's maximization problem is

$$\begin{aligned} \max_w \pi_1 + E\pi_2 &= \max_w \frac{3}{2}p_1 e(w,p_1) - 2w + \frac{1}{2}c(e(w,p_1)) \\ &= \max \Big\{ 0, \max_w \frac{3}{2}p_1 \tilde{e}(w,p_1) - 2w + \frac{1}{2}c(\tilde{e}(w,p_1)) \Big\}, \end{aligned}$$

where the 0 is the firm's outside-option payoff. The first-order condition gives

$$\left(\frac{3}{2}p_1 + \frac{1}{2}c'(\tilde{e}(w,p_1))\right)\frac{\partial \tilde{e}(w,p_1)}{\partial w} = 2.$$

Moreover, differentiating the equation defining the worker's solution $\tilde{e}(w,p_1)$, $\tilde{\pi}(w,\tilde{e};p_1) = \tilde{u}(w,\tilde{e})$, with respect to w gives

$$c'(\tilde{e}(w,p_1))\frac{\partial \tilde{e}(w,p_1)}{\partial w} + p_1\frac{\partial \tilde{e}(w,p_1)}{\partial w} = 2.$$

These two equations taken together imply $c'(\tilde{e}(w,p_1)) = p_1$, and so $\tilde{e}(w,p_1) = e_{\text{eff}}(p_1)$. The optimizing wage equates the surplus payoffs at the efficient level of effort and is given by equation (4).

Case 2. $\mu_A > \frac{1-2\sigma}{2}$. In this case, the solution to the worker's problem depends on whether $w > \overline{w}(p_1)$ or $w \leq \overline{w}(p_1)$. Hence the firm's maximization problem is now

$$\max\Big\{0, \underbrace{\max_{w\leq\overline{w}(p_1)} p_1\tilde{e}(w,p_1) - w}_{\text{(A)}}, \underbrace{\max_{w>\overline{w}(p_1)} p_1 e_{\text{I}} - w}_{\text{(B)}}\Big\}.$$

The candidate interior solution to sub-problem (A), w_{A}, is identical to the previous case with the solution being given by $w_{\text{A}} = \frac{c(e_{\text{eff}}(p_1)) + p_1 e_{\text{eff}}(p_1) - \pi_0 + u_0}{2}$.

As in the proof of Proposition 1, the maximand in (B) is strictly decreasing in w, but a candidate solution is that the worker's employment constraint binds with equality. However, even if there exists a solution to (B), $w_{\text{B}} > \overline{w}(p_1)$, the solution to (A) dominates it since, recalling that $\tilde{e}(\overline{w}(p_1),p_1) = e_{\text{I}}$,

$$p_1\tilde{e}(w_{\text{A}},p_1) - w_{\text{A}} \geq p_1\tilde{e}(\overline{w}(p_1),p_1) - \overline{w}(p_1) > p_1 e_{\text{I}} - w_{\text{B}}.$$

We conclude that the unique candidate solution is the solution to (A). Note that the solution to (A) is interior if and only if $w_{\text{A}} \leq \overline{w}(p_1)$, or equivalently, $e_{\text{I}} \geq \tilde{e}(w,p_1) = e_{\text{eff}}(p_1)$. Given equation (5), this is equivalent to $\mu_A \leq \frac{1-\sigma}{2}$, which is guaranteed by Assumption A$'$. Hence $(w_1^*, e_1^*) = (w_{\text{A}}, e_{\text{eff}}(p_1))$.

Finally, we check the firm's and the worker's employment constraints. The firm makes the wage offer whenever $\pi_1 + E\pi_2 \geq 0$. Note that

$$\begin{aligned}
\pi_1 + E\pi_2 &= \frac{3}{2}p_1e_1 - 2w_1 + \frac{1}{2}c(e_1) + \frac{1}{2}EM(p_2) \\
&= \frac{3}{2}p_1e_1 - \Big(c(e_1) + p_1e_1 - \pi_0 + u_0\Big) + \frac{1}{2}c(e_1) + \frac{1}{2}EM(p_2) \\
&= \frac{p_1e_1 - c(e_1) + EM(p_2)}{2} + \pi_0 - u_0 \\
&= \frac{M(p_1) + EM(p_2)}{2} + \pi_0 - u_0.
\end{aligned}$$

Therefore, the firm makes the wage offer if and only if

$$u_0 - \pi_0 \le \frac{M(p_1) + EM(p_2)}{2}.$$

The worker accepts the job offer whenever $U_1^E + EU_2^E \ge 0$. Note that

$$\begin{aligned} U_1^E &= w_1 - c(e_1) - (1-\sigma)u_0 = \frac{p_1 e_1 - c(e_1) - \pi_0 + u_0}{2} - (1-\sigma)u_0 \\ &= \frac{M(p_1) - \pi_0 + (2\sigma - 1)u_0}{2} \end{aligned}$$

and

$$\begin{aligned} EU_2^E &= \frac{EM(p_2) - \pi_1 + (2\sigma-1)u_1}{2} = \frac{EM(p_2) - (p_1 e_1 - w_1) + (2\sigma - 1)(w_1 - c(e_1))}{2} \\ &= \frac{2EM(p_2) - (p_1 e_1 - c(e_1) + \pi_0 - u_0) + (2\sigma - 1)(p_1 e_1 - c(e_1) - \pi_0 + u_0)}{4} \\ &= \frac{EM(p_2) - (1-\sigma)M(p_1) - \sigma\pi_0 + \sigma u_0}{2}. \end{aligned}$$

Summing, we have

$$U_1^E + EU_2^E = \frac{\sigma M(p_1) + EM(p_2) - (1+\sigma)\pi_0 + (3\sigma - 1)u_0}{2}.$$

Hence the worker accepts the offer if and only if

$$(1+\sigma)\pi_0 - (3\sigma - 1)u_0 \le \sigma M(p_1) + EM(p_2).$$

Both of these conditions are guaranteed by Assumption B′.

□

Proposition 4. *At the equilibrium described in Proposition 3: for $t = 1, 2$,*

1. *w_t^* and e_t^* are both increasing in p_t.*

2. *e_t^* does not depend on u_0 nor π_0.*

3. *w_t^* is increasing in u_0 and decreasing in π_0.*

Proof: The result has already been proven for $t = 2$ in Proposition 2. Since w_1^* is given by equation (4), it is increasing in p_1 and u_0, and decreasing in π_0. On the other hand, $e_1^* = e_{\text{eff}}(p_1)$, and so it is independent of both u_0 and π_0, and is increasing in p_1.

□

Lemma 2. *Under Assumption A, if $\lambda > 1$, then for any $p > 1$, there exists a $\overline{w}(p)$ such that for $w_0 < \overline{w}(p)$:*

1. *Effort responds more to wage cuts than to wage increases:* $\frac{\lim_{w \uparrow w_0} \frac{\partial e(w,p)}{\partial w}}{\lim_{w \downarrow w_0} \frac{\partial e(w,p)}{\partial w}} > 1.$

2. *Effort is more responsive to wage changes when effort is below the reference level of effort than when effort is below it:* $\frac{\lim_{e \uparrow e_0} \frac{\partial e(w,p)}{\partial w}}{\lim_{e \downarrow e_0} \frac{\partial e(w,p)}{\partial w}} > 1.$

If $w_0 \geq \overline{w}(p)$*, then* $e(w,p)$ *is constant in* w*.*

Proof: By an analogous argument as in the proof of Lemma 1, on $w < \overline{w}(p)$, $e(w,p) \equiv \tilde{e}(w,p)$, where $\tilde{e}(w,p)$ is defined by $\tilde{u}(w,\tilde{e}) = \tilde{\pi}(w,\tilde{e})$. Differentiating with respect to w:

$$\frac{\partial e}{\partial w} = \begin{cases} \frac{2}{p+c'(e)} & w \geq w_0, e \leq e_0, \\ \frac{2}{p+\lambda c'(e)} & w \geq w_0, e > e_0, \\ \frac{1+\lambda}{p+c'(e)} & w < w_0, e \leq e_0, \\ \frac{1+\lambda}{p+\lambda c'(e)} & w < w_0, e > e_0. \end{cases}$$

Since $\lambda > 1$ and $c' > 0$, the results of the lemma follow immediately. For $w_0 \geq \overline{w}(p)$, a similar argument as in the proof of Lemma 1 implies that $e(w,p)$ is constant in w.

□

Proposition 5. *Under Assumption A, if the firm hires the worker in equilibrium, then the equilibrium* (w^*, e^*) *is unique almost surely. Moreover:*

1. *If* $p \in \left(\underline{p}_{\text{w-rigid}}, \overline{p}_{\text{w-rigid}}\right)$*, then* $w^* = w_0$*,* $e^* > e_0$*, and* e^* *is strictly decreasing in* p*.*

2. *If* $p \in \left(\underline{p}_{\text{e-rigid}}, \overline{p}_{\text{e-rigid}}\right)$*, then* $e^* = e_0$*,* $w^* > w_0$*, and* w^* *is strictly increasing in* p*.*

3. *If* p *is outside the above ranges, then* w^* *and* e^* *are both strictly increasing in* p*.*

Proof: Consider the steady-state case: $c'(e_0) = p_0$. If $p = p_0$, then clearly $w^* = w_0$ and $e^* = e_0$.

For $p < p_0$, the firm has two options: $w < w_0$ or $w \geq w_0$. Consider first $w \geq w_0$. Since the worker chooses effort to equalize the surpluses, it must be that $e > e_0$ since $p < p_0$ and $w \geq w_0$. To see this, recall that the equalizing level of effort is given by

$$w - \theta c(e) - w_0 + \theta c(e_0) = pe - w - p_0 e_0 + w_0$$

for $\theta \in \{1, \lambda\}$, depending on whether or not $e > e_0$. Rearranging and using the fact that $w \geq w_0$ and $p_0 > p$,

$$\theta(c(e) - c(e_0)) = 2(w - w_0) + p_0 e_0 - pe > p_0(e_0 - e).$$

Since $c' > 0$, this can only hold for $e > e_0$. Now Lemma 2 implies that $\frac{\partial e}{\partial w} = \frac{2}{p+\lambda c'(e)}$. Note that the profit $\pi(w, e(w)) = pe - w$ is in fact decreasing in w since

$$\frac{\partial \pi}{\partial w} = p\frac{\partial e}{\partial w} - 1 = \frac{p - \lambda c'(e)}{p + \lambda c'(e)} < 0.$$

The final inequality follows from the fact that $c'(e) > c'(e_0) = p_0 > p$ and $\lambda \geq 1$. Thus the firm's solution on $w \geq w_0$ is $w^* = w_0$, and e^* is defined by

$$\lambda c(e^*) + pe^* = p_0 e_0 + \lambda c(e_0). \tag{6}$$

Turning to $w < w_0$, we now have $e < e_0$ using a similar argument as before. By Lemma 2, $\frac{\partial e}{\partial w} = \frac{1+\lambda}{p+c'(e)}$. Therefore, $\frac{\partial \pi}{\partial w} = \frac{\lambda p - c'(e)}{p+c'(e)}$. The first-order condition implies $c'(e^*) = \lambda p$. However, we require $e < e_0$, or equivalently, $c'(e) < c'(e_0) = p_0$. Therefore, for $p \in \left(\frac{p_0}{\lambda}, p_0\right)$, the maximization problem has no solution. If $p \leq \frac{p_0}{\lambda}$, then the solution is given by $c'(e^*) = \lambda p$, and w^* is the wage such that the surpluses are equalized:

$$w^* = \frac{pe^* - p_0 e_0 + (1+\lambda)w_0 + c(e^*) - c(e_0)}{1+\lambda}.$$

Since $c'(e^*) = \lambda p$, we know that $e^* < e_0$, and thus $w^* = w_0 + \frac{pe^* - p_0 e_0 + c(e^*) - c(e_0)}{1+\lambda} < w_0$.

We conclude as follows: in the steady-state case with $p \in \left(\frac{p_0}{\lambda}, p_0\right)$, $w^* = w_0$ and $e^* > e_0$ is defined by equation (6). Analysis of equation (6) shows that e^* must be decreasing in p since the right-hand side is constant and the left-hand side is strictly increasing in e. For $p < \frac{p_0}{\lambda}$, $e^* < e_0$ and $w^* < w_0$ are strictly increasing in p. Denoting profit when the firm sets that wage by $\pi_{<w_0}(p)$ and profit when the firm sets wage w_0 by $\pi_{w_0}(p)$, note that because both functions are continuous in p, $\pi_{w_0}\left(\frac{p_0}{\lambda}\right) = \pi_{<w_0}\left(\frac{p_0}{\lambda}\right)$. The firm is therefore indifferent, and thus the equilibrium is non-unique, when the price realization is exactly $\frac{p_0}{\lambda}$.

For $p > p_0$, the firm again has the same two options: $w < w_0$ or $w \geq w_0$. Consider first $w < w_0$. As before, it must be the case that $e < e_0$. Therefore, $\frac{\partial e}{\partial w} = \frac{1+\lambda}{p+c'(e)}$. Hence, $\frac{\partial \pi}{\partial w} = \frac{p\lambda - c'(e)}{p+c'(e)} > 0$ since $e < e_0$ implies $c'(e) < c'(e_0) = p_0 < p$ and $\lambda \geq 1$. That is, profit is strictly increasing in wage and so there is no solution for $w < w_0$.

Now consider $w \geq w_0$. We analyze three subcases: (a) $w = w_0$, (b) $w > w_0$ and $e \leq e_0$, and (c) $w > w_0$ and $e > e_0$. For (a), since $p > p_0$, it must be that $e < e_0$ since the worker chooses effort to equalize the surplus payoffs. Thus for both (a) and (b), we have that $\frac{\partial \pi}{\partial w} = \frac{p - c'(e)}{p+c'(e)} > 0$ since $c'(e) < c'(e_0) = p_0 < p$. But this implies that $e > e_0$, a contradiction. Thus we know that if $w \geq w_0$, then we are in case (c).

For (c), $\frac{\partial \pi}{\partial w} = \frac{p-\lambda c'(e)}{p+\lambda c'(e)}$. The firm's first-order condition thus implies $c'(e) = \frac{p}{\lambda}$. Since we are on the domain $e > e_0$, or equivalently, $c'(e) > c'(e_0) = p_0$, for $p < \lambda p_0$, we have a corner solution, $e^* = e_0$. Since the wage is such that the surpluses are equalized, it must be increasing in p since profit is increasing in p, and effort and thus utility are otherwise constant. For $p \geq \lambda p_0$, the solution is given by $c'(e^*) = \frac{p}{\lambda}$ and the wage w^* that equalizes the surpluses:

$$w^* = \frac{pe^* - p_0 e_0 + 2w_0 + \lambda c(e^*) - \lambda c(e_0)}{2}.$$

Clearly e^* and w^* are both increasing in p since $c' > 0$ and $c'' > 0$.

Now consider the recent-increase case: $p_0 = \lambda c'(e_0)$. Define $p_{00} \equiv \frac{p_0}{\lambda}$. Then the previous analysis applies to $p_{00} = c'(e_0)$. That is, for $p \in \left(\frac{p_{00}}{\lambda}, p_{00}\right) = \left(\frac{p_0}{\lambda^2}, \frac{p_0}{\lambda}\right)$, $w^* = w_0$, and $e^* > e_0$ is decreasing in p. For $p \in (p_{00}, \lambda p_{00}) = \left(\frac{p_0}{\lambda}, p_0\right)$, $e^* = e_0$, and $w^* \geq w_0$ is increasing in p. For all other p, w^* and e^* are increasing in p.

Finally, the recent-decrease case: $\lambda p_0 = c'(e_0)$. Similarly to above, define $p_{00} \equiv \lambda p_0$, and apply the steady-state-case result to $p_{00} = c'(e_0)$. That is, for $p \in \left(\frac{p_{00}}{\lambda}, p_{00}\right) = (p_0, \lambda p_0)$, $w^* = w_0$, and $e^* > e_0$ is decreasing in p. For $p \in (p_{00}, \lambda p_{00}) = (\lambda p_0, \lambda^2 p_0)$, $e^* = e_0$, and $w^* \geq w_0$ is increasing in p. For all other p, w^* and e^* are increasing in p.

□

Proposition 6. *Under Assumptions A and B, if the worker is not loss averse ($\lambda = 1$), then for any $p > 1$, the equilibrium transaction is UPE and MPE.*

Proof: By Proposition 1, $e^* = \tilde{e}(w^*, e^*) = e_{\text{eff}}(p)$. Therefore, the equilibrium is MPE. Since we are in the case of no loss aversion,

$$U^E = \mu(w - c(e)) + (1 - \mu)(pe - w) + \sigma u_0,$$

where $\mu \in \{\mu_D, \mu_A\}$. Thus for each μ,

$$\frac{\partial U/\partial w}{\partial U/\partial e} = \frac{\mu - (1 - \mu)}{-\mu c'(e) + (1 - \mu)p}.$$

Since $c'(e^*) = c'(e_{\text{eff}}(p)) = p$,

$$\lim_{(w,e)\to(w^*,e^*)} \frac{\partial U/\partial w}{\partial U/\partial e} = \frac{2\mu - 1}{(1 - 2\mu)p} = \frac{-1}{p}.$$

Moreover, since $\pi = pe - w$,

$$\frac{\partial \pi/\partial w}{\partial \pi/\partial e} = \frac{-1}{p}.$$

Since $\frac{\partial U/\partial w}{\partial U/\partial e} = \frac{\partial \pi/\partial w}{\partial \pi/\partial e}$, (w^*, e^*) is UPE.

□

Proposition 7. *Under Assumption A, if the worker is loss averse ($\lambda > 1$) and the firm hires the worker in equilibrium, then the equilibrium transaction is UPE if and only if* $p \notin \left(\underline{p}_{\text{w-rigid}}, \overline{p}_{\text{w-rigid}}\right) \cup \left(\underline{p}_{\text{e-rigid}}, \overline{p}_{\text{e-rigid}}\right)$. *The equilibrium transaction is also MPE if and only if $p = \overline{p}_{\text{w-rigid}}$, or equivalently, if and only if the equilibrium transaction (w^*, e^*) satisfies $w^* = w_0$ and $e^* = e_0$.*

Proof: We first show that (w^*, e^*) is not UPE for $p \in (\underline{p}_{\text{w-rigid}}, \overline{p}_{\text{w-rigid}}) = \left(\frac{p_{00}}{\lambda}, p_{00}\right)$. Proposition 5 implies that $w^* = w_0$, $e^* > e_0$, and $c'(e^*) > c'(e_0) = p_{00} > p$. Let $\Delta > 0$ be small enough so that $e' = e^* - \Delta > e_0$ and define $(w', e') = (w^* - p\Delta, e^* - \Delta)$ so that

$$\pi(w', e') = pe' - w' = p(e^* - \Delta) - (w^* - p\Delta) = pe^* - w^* = \pi(w^*, e^*).$$

Since c is convex and $c'(e') > c'(e_0) > p$,

$$c(e^*) = c(e' + \Delta) \geq c(e') + c'(e')\Delta > c(e') + p\Delta.$$

Therefore, the worker's change in wage is $w' - w^* = -p\Delta$, and the change in the cost of effort is $c(e') - c(e^*) < -p\Delta$. It follows that $u(w', e') - u(w^*, e^*) > 0$. Since the firm's profit is unchanged, it must be that $U(w', e') > U(w^*, e^*)$. Thus (w^*, e^*) is not UPE.

We now show that (w^*, e^*) is not UPE for $p \in (\underline{p}_{\text{e-rigid}}, \overline{p}_{\text{e-rigid}}) = (p_{00}, \lambda p_{00})$. Proposition 5 implies that $e^* = e_0$ and $w^* > w_0$. Let $\Delta > 0$ be small enough so that $c'(e^* + \Delta) < p$ (this is possible since $c'(e^*) = p_{00} < p$ and c' is continuous), and define $(w', e') = (w^* + p\Delta, e^* + \Delta)$ so that $\pi(w', e') = \pi(w^*, e^*)$. Again, since c is convex,

$$c(e^*) = c(e' - \Delta) > c(e') - c'(e')\Delta > c(e') - p\Delta$$

since $c'(e') < p$. Thus, the worker's change in wage is $w' - w^* = p\Delta$, and his change in the cost of effort is $c(e') - c(e^*) < p\Delta$. We then have that $u(w', e') - u(w^*, e^*) > 0$. Since the firm's profit is unchanged, it must be that $U(w', e') > U(w^*, e^*)$. Thus (w^*, e^*) is not UPE.

There are three more cases to consider: (1) $p > \overline{p}_{\text{e-rigid}}$, (2) $p < \underline{p}_{\text{w-rigid}}$, and (3) $p = \overline{p}_{\text{w-rigid}}$. We analyze each in turn.

Case (1). From the proof of Proposition 5, we know that in this case, $w^* > w_0$, $e^* > e_0$, and $c'(e^*) = p/\lambda$. Therefore,

$$\frac{\partial U/\partial w}{\partial U/\partial e} = \frac{2\mu - 1}{-\mu\lambda c'(e^*) + p(1-\mu)} = \frac{2\mu - 1}{-\mu p + p(1-\mu)} = -\frac{1}{p}.$$

Since (as in the proof of Proposition 6) $\frac{\partial\pi/\partial w}{\partial\pi/\partial e} = -\frac{1}{p}$ as well, (w^*, e^*) is UPE.

Case (2). From the proof of Proposition 5, we know that in this case, $w^* < w_0$, $e^* < e_0$, and $c'(e^*) = \lambda p$. Therefore,

$$\frac{\partial U/\partial w}{\partial U/\partial e} = \frac{(1+\lambda)\mu - 1}{-\mu c'(e^*) + p(1-\mu)} = \frac{(1+\lambda)\mu - 1}{-\mu\lambda p + p(1-\mu)} = -\frac{1}{p}.$$

Thus, as before, (w^*, e^*) is UPE.

Case (3). From the proof of Proposition 5, we know that in this case, $w^* = w_0$, $e^* = e_0$, and $c'(e_0) = p$. For contradiction, suppose there exists (w', e') that utility-Pareto dominates (w^*, e^*). Define Δ_e and Δ_w by $e' = e_0 + \Delta_e$ and $w' = w_0 + \Delta_w$. Since $\pi(w', e') \geq \pi(w_0, e_0)$, we know that $\Delta_w \leq p\Delta_e$, and since (w', e') is a utility-Pareto improvement, we know that $\text{sgn}(\Delta_w) = \text{sgn}(\Delta_e)$.

Below, we will use the following result: Since c is convex,

$$c(e') = c(e_0 + \Delta_e) \geq c(e_0) + c'(e_0)\Delta_e = c(e_0) + p\Delta_e.$$

There are three sub-cases to consider: (a) $\Delta_e > 0$, (b) $\Delta_e < 0$, and (c) $\Delta_e = 0$. Starting with sub-case (a), the worker is experiencing a loss in the effort domain since $\Delta_e > 0$ and a gain in the wage domain since $\Delta_w > 0$. Thus,

$$\begin{aligned}\tilde{u}(w', e') - \tilde{\pi}(w', e') &= 2w' - 2w_0 - pe' + pe_0 - \lambda c(e') + \lambda c(e_0) \\ &\leq 2(w_0 + \Delta_w) - 2w_0 - p(e_0 + \Delta_e) + pe_0 - \lambda(c(e_0) + p\Delta_e) + \lambda c(e_0) \\ &= 2\Delta_w - p\Delta_e - \lambda p\Delta_e \leq (2 - (1+\lambda))p\Delta_e < 0,\end{aligned}$$

and therefore the worker is in the region of disadvantageous unfairness. Dropping the u_0 and π_0 terms in the worker's utility,

$$\begin{aligned}U(w', e') &= \mu_D\left(w' - \lambda c(e')\right) + (1-\mu_D)\left(pe' - w'\right) \\ &= (2\mu_D - 1)w' - \lambda\mu_D c(e') + (1-\mu_D)pe' \\ &\leq (2\mu_D - 1)(w_0 + \Delta_w) - \lambda\mu_D(c(e_0) + p\Delta_e) + (1-\mu_D)p(e_0 + \Delta_e) \\ &< (2\mu_D - 1)w_0 - \mu_D c(e_0) + (1-\mu_D)pe_0 + (2\mu_D - 1)\Delta_w + (1 - \mu_D - \lambda\mu_D)p\Delta_e \\ &< (2\mu_D - 1)w_0 - \mu_D c(e_0) + (1-\mu_D)pe_0 \\ &= U(w_0, e_0).\end{aligned}$$

But this contradicts the hypothesis that (w', e') utility-Pareto dominates (w_0, e_0).

Now consider sub-case (b), $\Delta_e < 0$. Since $\Delta_w < 0$, the worker is experiencing a loss in the wage domain. Since $c(e') - c(e_0) \geq p\Delta_e \geq \Delta_w$, we know that $u(w', e') - u(w_0, e_0) = \Delta_w - (c(e') - c(e_0)) \leq$

0, with strict inequality if $\Delta_w < p\Delta_e$. Suppose first that $\Delta_w = p\Delta_e$. Then $\pi(w', e') = \pi(w_0, e_0)$, and so $\tilde{\pi}(w', e') = \tilde{\pi}(w_0, e_0)$. Thus, for $\mu \in \{\mu_A, \mu_D\}$,

$$\begin{aligned} U(w', e') &= \mu\tilde{u}(w', e') + (1-\mu)\tilde{\pi}(w', e') + \sigma u_0 \\ &= \mu(u(w', e') - u_0) + (1-\mu)\tilde{\pi}(w_0, e_0) + \sigma u_0 \\ &\leq \mu(u(w_0, e_0) - u_0) + (1-\mu)\tilde{\pi}(w_0, e_0) + \sigma u_0 \\ &= \mu\tilde{u}(w_0, e_0) + (1-\mu)\tilde{\pi}(w_0, e_0) + \sigma u_0 = U(w_0, e_0). \end{aligned}$$

Therefore (w', e') cannot utility-Pareto dominate (w_0, e_0), a contradiction. Now suppose $\Delta_w < p\Delta_e$. Then since $\pi(w', e') > \pi(w_0, e_0)$, $u(w', e') < u(w_0, e_0)$, and (w_0, e_0) is a surplus-equalizing transaction, it must be that the worker is in the disadvantageously-unfair region. Furthermore, $\pi(w', e') > \pi(w_0, e_0)$ implies that $\tilde{\pi}(w', e') > \tilde{\pi}(w_0, e_0)$. Since $\mu_D \geq 1$ (from A(i)),

$$\begin{aligned} U(w', e') &= \mu_D\tilde{u}(w', e') + (1-\mu_D)\tilde{\pi}(w', e') + \sigma u_0 \\ &\leq \mu_D(u(w', e') - u_0) + (1-\mu_D)\tilde{\pi}(w_0, e_0) + \sigma u_0 \\ &< \mu_D(u(w_0, e_0) - u_0) + (1-\mu_D)\tilde{\pi}(w_0, e_0) + \sigma u_0 \\ &= \mu_D\tilde{u}(w_0, e_0) + (1-\mu_D)\tilde{\pi}(w_0, e_0) + \sigma u_0 = U(w_0, e_0). \end{aligned}$$

Again, this contradicts the hypothesis that (w', e') utility-Pareto dominates (w_0, e_0).

Last, we turn to sub-case (c), $\Delta_e = 0$. Since $\Delta_w \leq p\Delta_e$, we know that either $\Delta_w < 0$ or $\Delta_w = 0$. But the latter is impossible because then profit and utility would be unchanged (contradicting UPE), so $\Delta_w < 0$. Since $\Delta_e = 0$ and $\Delta_w < 0$, the worker is in the region of disadvantageous unfairness. But then a decrease in the wage reduces utility, so $U(w', e') < U(w_0, e_0)$, contradicting the hypothesis that (w', e') utility-Pareto dominates (w_0, e_0). We conclude that in case (3), $(w^*, e^*) = (w_0, e_0)$ is UPE.

Finally, we consider when the equilibrium is MPE, i.e., when $c'(e) = p$. According to Proposition 5, this only occurs when $p = p_{00}$ and $(w^*, e^*) = (w_0, e_0)$.

□